Problem 1.
Let $\mathbb{P}$ be the set of all prime numbers. Find all functions $f : \mathbb{P} \to \mathbb{P}$ such that

$$f(p)^{f(q)} + q^p = f(q)^{f(p)} + p^q$$

holds for all $p, q \in \mathbb{P}$.

Proposed by Albania

Solution. Obviously, the identical function $f(p) = p$ for all $p \in \mathbb{P}$ is a solution. We will show that this is the only one.

First we will show that $f(2) = 2$. Taking $q = 2$ and p any odd prime number, we have

$$f(p)^{f(2)} + 2^p = f(2)^{f(p)} + p^2.$$

Assume that $f(2) \neq 2$. It follows that $f(2)$ is odd and so $f(p) = 2$ for any odd prime number p.

Taking any two different odd prime numbers p, q we have

$$2^2 + q^p = 2^2 + p^q \quad \Rightarrow \quad p^q = q^p \quad \Rightarrow p = q,$$

contradiction. Hence, $f(2) = 2$.

So for any odd prime number p we have

$$f(p)^2 + 2^p = 2^{f(p)} + p^2.$$

Copy this relation as

$$2^p - p^2 = 2^{f(p)} - f(p)^2. \tag{1}$$

Let T be the set of all positive integers greater than 2, i.e. $T = \{3, 4, 5, \ldots\}$. The function $g : T \to \mathbb{Z}$, $g(n) = 2^n - n^2$, is strictly increasing, i.e.

$$g(n+1) - g(n) = 2^n - 2n - 1 > 0 \tag{2}$$

for all $n \in T$. We show this by induction. Indeed, for $n = 3$ it is true, $2^3 - 2 \cdot 3 - 1 > 0$. Assume that $2^k - 2k - 1 > 0$. It follows that for $n = k + 1$ we have

$$2^{k+1} - 2(k+1) - 1 = (2^k - 2k - 1) + (2^k - 2) > 0$$

for any $k \geq 3$. Therefore, (2) is true for all $n \in T$.

As consequence, (1) holds if and only if $f(p) = p$ for all odd prime numbers p, as well as for $p = 2$.

Therefore, the only function that satisfies the given relation is $f(p) = p$, for all $p \in \mathbb{P}$.
□

Problem 2.
Let a, b, c be real numbers, such that $0 \leq a \leq b \leq c$ and $a+b+c = ab+bc+ca > 0$. Prove that $\sqrt{bc}(a+1) \geq 2$. Find all triples (a, b, c) for which equality holds.

Proposed by Romania

Solution. Let $a+b+c = ab+bc+ca = k$. Since $(a+b+c)^2 \geq 3(ab+bc+ca)$, we get that $k^2 \geq 3k$. Since $k > 0$, we obtain that $k \geq 3$.

We have $bc \geq ca \geq ab$, so from the above relation we deduce that $bc \geq 1$.

By AM-GM, $b+c \geq 2\sqrt{bc}$ and consequently $b+c \geq 2$. The equality holds iff $b = c$.

The constraint gives us

$$a = \frac{b+c-bc}{b+c-1} = 1 - \frac{bc-1}{b+c-1} \geq 1 - \frac{bc-1}{2\sqrt{bc}-1} = \frac{\sqrt{bc}(2-\sqrt{bc})}{2\sqrt{bc}-1}.$$

For $\sqrt{bc} = 2$ condition $a \geq 0$ gives $\sqrt{bc}(a+1) \geq 2$ with equality iff $a = 0$ and $b = c = 2$.

For $\sqrt{bc} < 2$, taking into account the estimation for a, we get

$$a\sqrt{bc} \geq \frac{bc(2-\sqrt{bc})}{2\sqrt{bc}-1} = \frac{bc}{2\sqrt{bc}-1}(2-\sqrt{bc}).$$

Since $\frac{bc}{2\sqrt{bc}-1} \geq 1$, with equality for $bc = 1$, we get $\sqrt{bc}(a+1) \geq 2$ with equality iff $a = b = c = 1$.

For $\sqrt{bc} > 2$ we have $\sqrt{bc}(a+1) > 2(a+1) \geq 2$.

The proof is complete.

The equality holds iff $a = b = c = 1$ or $a = 0$ and $b = c = 2$. $\square$

Problem 3.
Let ABC be an acute scalene triangle. Let X and Y be two distinct interior points of the segment BC such that $\angle CAX = \angle YAB$. Suppose that:

1) K and S are the feet of perpendiculars from B to the lines AX and AY respectively;
2) T and L are the feet of perpendiculars from C to the lines AX and AY respectively.

Prove that KL and ST intersect on the line BC.

Proposed by Greece

Solution. Denote $\phi = \widehat{XAB} = \widehat{YAC}, \alpha = \widehat{CAX} = \widehat{BAY}$. Then, because the quadrilaterals ABSK and ACTL are cyclic, we have

$$\widehat{BSK} + \widehat{BAK} = 180^\circ = \widehat{BSK} + \phi = \widehat{LAC} + \widehat{LTC} = \widehat{LTC} + \phi,$$

so, due to the 90-degree angles formed, we have $\widehat{KSL} = \widehat{KTL}$. Thus, KLST is cyclic.

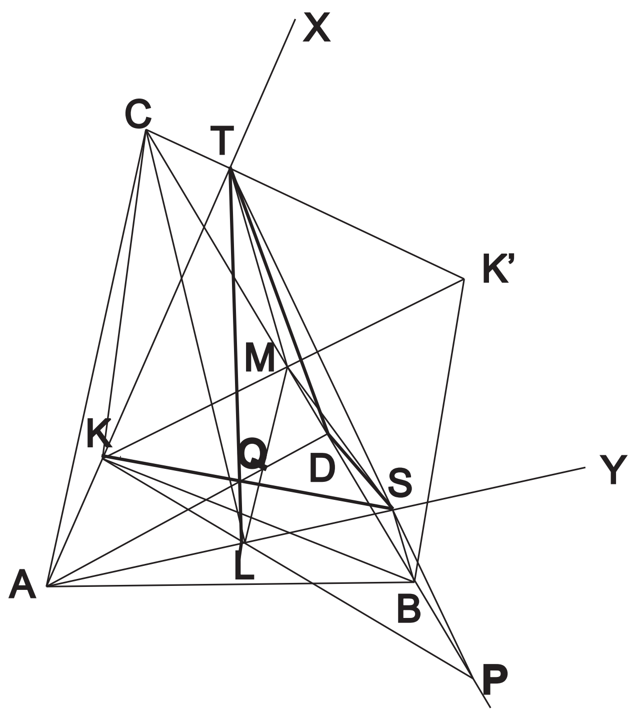

Figure 1: G6

Consider M to be the midpoint of BC and K' to be the symmetric point of K with respect to M. Then, $BKCK'$ is a parallelogram, and so $BK \| CK'$. But $BK \| CT$, because they are both perpendicular to AX. So, K' lies on CT and, as $\widehat{KTK'} = 90$ and M is the midpoint of KK', $MK = MT$. In a similar way, we have that $MS = ML$. Thus, the center of $(KLST)$ is M.

Consider D to be the foot of altitude from A to BC. Then, D belongs in both $(ABKS)$ and $(ACLT)$. So,

$$\widehat{ADT} + \widehat{ACT} = 180^\circ = \widehat{ABS} + \widehat{ADS} = \widehat{ADT} + 90^\circ - \alpha = \widehat{ADS} + 90^\circ - \alpha,$$

and AD is the bisector of $\widehat{SDT}$.

Because DM is perpendicular to AD, DM is the external bisector of this angle, and, as $MS = MT$, it follows that $DMST$ is cyclic. In a similar way, we have that $DMLK$ is also cyclic.

So, we have that ST, KL and DM are the radical axes of these three circles, $(KLST)$, $(DMST)$, $(DMKL)$. These lines are, therefore, concurrent, and we have proved the desired result. $\square$

Problem 4.
A grid consists of all points of the form (m, n) where m and n are integers with $|m| \leqslant 2019$, $|n| \leqslant 2019$ and $|m| + |n| < 4038$. We call the points (m, n) of the grid with either $|m| = 2019$ or $|n| = 2019$ the *boundary points*. The four lines $x = \pm 2019$ and $y = \pm 2019$ are called *boundary lines*. Two points in the grid are called *neighbours* if the distance between them is equal to 1.

Anna and Bob play a game on this grid.

Anna starts with a token at the point $(0, 0)$. They take turns, with Bob playing first.

1) On each of his turns, Bob *deletes* at most two boundary points on each boundary line.

2) On each of her turns, Anna makes exactly three *steps*, where a *step* consists of moving her token from its current point to any neighbouring point which has not been deleted.

As soon as Anna places her token on some boundary point which has not been deleted, the game is over and Anna wins.

Does Anna have a winning strategy?

Proposed by Cyprus

Solution. Anna does not have a winning strategy. We will provide a winning strategy for Bob. It is enough to describe his strategy for the deletions on the line $y = 2019$.

Bob starts by deleting $(0, 2019)$ and $(-1, 2019)$. Once Anna completes her turn, he deletes the next two available points on the left if Anna decreased her x-coordinate, the next two available points on the right if Anna increased her x-coordinate, and the next available point to the left and the next available point to the right if Anna did not change her x-coordinate. The only exception to the above rule is on the very first time Anna decreases x by exactly 1. In that turn, Bob deletes the next available point to the left and the next available point to the right.

Bob's strategy guarantees the following: If Anna makes a sequence of steps reaching $(-x, y)$ with $x > 0$ and the exact opposite sequence of steps in the horizontal direction reaching (x, y), then Bob deletes at least as many points to the left of $(0, 2019)$ in the first sequence than points to the right of $(0, 2019)$ in the second sequence.

So we may assume for contradiction that Anna wins by placing her token at $(k, 2019)$ for some $k > 0$.

Define $\Delta = 3m - (2x + y)$, where m is the total number of points deleted by Bob to the right of $(0, 2019)$, and (x, y) is the position of Anna's token.

For each sequence of steps performed first by Anna and then by Bob, Δ does not decrease. This can be seen by looking at the following table exhibiting the changes in $3m$ and $2x + y$. We have excluded the cases where $2x + y < 0$.

Turn	(0,3)	(1,2)	(-1,2)	(2,1)	(0,1)	(3,0)	(1,0)	(2,-1)	(1,-2)
m	1	2	0 (or 1)	2	1	2	2	2	2
$3m$	3	6	0 (or 3)	6	3	6	6	6	6
$2x + y$	3	4	0	5	1	6	2	3	0

The table also shows that, if in this sequence of turns Anna changes y by $+1$ or -2, then Δ is increased by 1. Also, if Anna changes y by $+2$ or -1, then the first time this happens Δ is increased by 2. (This also holds if her turn is $(0,-1)$ or $(-2,-1)$, which are not shown in the table.)

Since Anna wins by placing her token at $(k, 2019)$ we must have $m \leqslant k-1$ and $k \leqslant 2018$. So at that exact moment we have:

$$\Delta = 3m - (2k + 2019) = k - 2022 \leqslant -4.$$

So in her last turn she must have decreased Δ by at least 4. So her last turn must have been $(1,2)$ or $(2,1)$, which give a decrease of 4 and 5 respectively. (It could not be $(3,0)$ because then she must have already won. Also she could not have done just one or two steps in her last turn since this is not enough for the required decrease in Δ.)

If her last turn was $(1,2)$, then just before doing it we had $y = 2017$ and $\Delta = 0$. This means that in one of her turns the total change in y was not 0 mod 3. However, in that case we have seen that $\Delta > 0$, a contradiction.

If her last turn was $(2,1)$, then just before doing it we had $y = 2018$ and $\Delta = 0$ or $\Delta = 1$. So she must have made at least two turns with the change of y being $+1$ or -2 or at least one step with the change of y being $+2$ or -1. In both cases, consulting the table, we get an increase of at least 2 in Δ, a contradiction.

Note 1: If Anna is allowed to make **at most** three steps at each turn, then she actually has a winning strategy.

Note 2: If 2019 is replaced by $N > 1$, then Bob has a winning strategy if and only if $3 \mid N$. $\square$